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The transcendental method in the theory of neutron slowing down

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Abstract. The transcendental method for finding the exact analytical closed-form solution to the linear unidimensional integral equation of neutron slowing down with energy-dependent cross-section in an infinite homogeneous medium is studied in some detail. An original method of genesis of the isomorphic integral form of the process, and genesis of the general form of the analytical solution is applied. This, together with the exact solution of the transcendental equation of order one also determined, constitutes the exact solution of the problem. The numerical results obtained for different magnitudes of absorption rates and for different moderator masses show agreement with more conventional solutions, like those of Teichmann and Sengupta. The conditions for the existence of the exact solutions are discussed.

1. Introduction

The transcendental method for obtaining meaningful results from analytical study of the time-independent slowing down equation for neutrons in an infinite moderator consisting of heavier nuclei with isotropic elastic scattering and with energy-dependent cross-section is an important one.

The present paper researches possibilities for an analytical solution of the linear integral unidimensional equation of slowing down of neutrons because the former methods (Davison 1960, 1954, Placzek 1946, Verde 1947, Keane 1961, Dawn 1972, Barnett 1974 and Sengupta *et al* 1974) cannot express the solution by an analytical closed form, and only the collision density over the first few collision intervals can be derived exactly.

The slowing down equation with energy-dependent cross sections has not yet been solved. Bednarz (1961) obtained an analytical solution by applying a method which is mathematically quite difficult. Dawn (1976) obtained analytical solutions of special cases of this equation.

Under the assumptions that the nuclei are considered at rest before the collision and that the scattering is isotropic in the centre-of-mass system, the well-known time-independent neutron slowing down equation for the total collision density, $F(u)$, in the lethargy domain is

$$F(u)=(1+\beta)\int_{u-a}^{u} \mathrm{d}u'\, \mathrm{e}^{u'-u}\Psi_{\mathrm{s}}(u')F(u')+\delta(u) \qquad (1)$$

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where the symbols have the following meaning:

$$\alpha=((A-1)/(A+1))^2 \qquad A=M/m$$

where M is the mass of the nucleus, and m is the neutron mass;

$$\beta=\alpha/(1-\alpha)$$

$$a=-\ln(\alpha)$$

$$\Psi_s(u)=\Sigma_s(u)/\Sigma_t(u)$$

$$\Psi_a(u)=1-\Psi_s(u)$$

$\Sigma_t(u)$ is the total cross section. The subscripts s and a refer to the elastic and absorption collision, respectively; $\delta(u)$ is the Dirac delta function and denotes the energy point source.

In this presentation, equation (1) has been analytically solved. A Laplace transform has been applied for the genesis of the isomorphic model of the process. For the generated isomorphic model of the process a solution has been proposed in the exponential form. An unknown exponent in the suggested solution represents analytical solutions of a transcendental equation of order one which is analytically solvable.

An example for analytical solution of the corresponding transcendental equation is given in section 8.

The analytical solutions which have been obtained, apart from theoretical values in academic discussions, have also a practical application since they are far simpler, effective and easy to examine than more conventional solutions.

2. Genesis of the isomorphic model of the process

Applying a Laplace transform it is possible to modify integral equation (1) into the isomorphic model of the process which is described by Volterra's integral equation:

$$F(u)=f_{0\delta}(u)-\int_0^u \mathrm{d}u'\, f_0(u-u')\Psi_a(u')F(u') \tag{2}$$

where

$$f_{0\delta}(u)=f_0(u)+\delta(u). \tag{3}$$

$f_0(u)$ is the well-known Teichmann (1960) series for collision density, and

$$f_0(u)=(1+\beta)\,\mathrm{e}^{\beta u}\sum_{n=0}^{[u/a]}(-1)^n\beta^n\,\mathrm{e}^{-na\beta}(u-na)^n/n!$$
$$+\mathrm{e}^{\beta u}\sum_{n=1}^{[u/a]}(-1)^n\beta^n\,\mathrm{e}^{-na\beta}(u-na)^{n-1}/(n-1)! \tag{4}$$

where $[u/a]$ denotes the greatest integer less than or equal to u/a.

Proof. The Laplace transform of the integral equation (1)

$$\varphi(s)=(1+\beta)\varphi(s)\frac{1-\mathrm{e}^{-a(s+1)}}{s+1}-(1+\beta)P(s)\frac{1-\mathrm{e}^{-a(s+1)}}{s+1}+1 \tag{5}$$

where

$$\varphi(s) = \int_0^\infty F(u)\, e^{-su}\, du$$

$$P(s) = \int_0^\infty F(u)\Psi_a(u)\, e^{-su}\, du$$

may be written in the form

$$\varphi(s) = \varphi_{0\delta}(s) - P(s)\varphi_0(s) \tag{6}$$

where

$$\varphi_{0\delta}(s) = \frac{s+1}{s-\beta+\beta\, e^{-as}} = \int_0^\infty f_{0\delta}(u)\, e^{-su}\, du$$

and

$$\varphi_0(s) = \frac{1+\beta-\beta\, e^{-as}}{s-\beta+\beta\, e^{-as}} = \int_0^\infty f_0(u)\, e^{-su}\, du.$$

Now, by using the convolution theorem (Murray 1965) equation (6) in the lethargy domain takes the form

$$F(u) = f_{0\delta}(u) - \int_0^u f_0(u-u')\Psi_a(u')F(u')\, du' \tag{7}$$

which coincides with equation (2).

From equation (1), we have for $u=0$

$$F^*(0) + \delta(0) = (1+\beta)\Psi_s(0) + \delta(0) \tag{8}$$

and from equation (7)

$$F^*(0) + \delta(0) = f_{0\delta}(0) - f_0(0)\Psi_a(0)$$

or

$$F^*(0) + \delta(0) = (1+\beta)\Psi_s(0) + \delta(0) \tag{9}$$

since

$$f_{0\delta}(0) = f_0(0) + \delta(0) \qquad \Psi_s(0) = 1 - \Psi_a(0)$$

where we define

$$F^*(u) \equiv F(u) - \delta(u).$$

Obviously, equations (8) and (9) are identical. This is, of course, expected since equation (7) is an equivalent form of equation (1). Further, from equation (7) we have

$$F(u) = f_{0\delta}(u) \qquad \text{for } \Psi_a(u) = 0.$$

Now we have completed our proof. □

3. Some important properties of the function $f_{0\delta}(u)$

The interesting function $f_{0\delta}(u)$ is an analytical solution of equation (1) in the particular case for $\Psi_a(u)=0$. Namely, the function $f_{0\delta}(u)$ is given by

$$f_{0\delta}(u)\,e^{u}=(1+\beta)\int_{u-a}^{u} f_{0\delta}(u')\,e^{u'}\,du'+\delta(u)\,e^{u} \tag{10}$$

or

$$f_{0\delta}(u)-\beta\int_{u-a}^{u} f_{0\delta}(u')\,du'=1+\delta(u). \tag{10'}$$

Equation (10′) is an equivalent form of equation (10).

The function $f_0(u)$ satisfies the following integral equation:

$$f_0(u)-\beta\int_{u-a}^{u} f_0(u')\,du'=1. \tag{11}$$

The Laplace transform of equation (10) (or (10′)) takes the form

$$\varphi_{0\delta}(s)=\frac{s+1}{s-\beta+\beta\,e^{-as}}$$

and

$$\varphi_{0\delta}(s+\beta)=\frac{s+1+\beta}{s(1+\beta\,e^{-a\beta}\,e^{-as}/s)}. \tag{12}$$

For $s>\beta$, this equation can be rewritten as

$$\varphi_{0\delta}(s+\beta)=(1+\beta)\sum_{n=0}^{\infty}(-1)^n\beta^n\,e^{-na\beta}\,e^{-nas}/s^{n+1}+\sum_{n=0}^{\infty}(-1)^n\beta^n\,e^{-na\beta}\,e^{-nas}/s^{n}. \tag{13}$$

Of course, a condition under which the result is valid is that the series (13) is convergent for $s>\beta$. Then, we can invert term by term to obtain, in the lethargy domain

$$f_{0\delta}(u)\,e^{\beta u}=(1+\beta)\sum_{n=0}^{[u/a]}(-1)^n\beta^n\,e^{-na\beta}(u-na)^n/n!+\delta(u)$$
$$+\sum_{n=1}^{[u/a]}(-1)^n\beta^n\,e^{-na\beta}(u-na)^{n-1}/(n-1)! \tag{14}$$

Since (Murray 1965)

$$e^{-nas}/s^{n+1}\to(u-na)^nH(u-na)/n!$$
$$e^{-nas}/s^{n}\to(u-na)^{n-1}H(u-na)/(n-1)!$$

Where Heaviside's unit function is defined as

$$H(u-na)=\begin{cases}0 & \text{for } u>na\\ 1 & \text{for } u<na.\end{cases}$$

From the relation in equation (14) $f_{0\delta}(u)$ can be expressed as

$$f_{0\delta}(u)=(1+\beta)\,e^{\beta u}R(u)+e^{\beta u}R'(u)+\delta(u) \tag{15}$$

where the series $R(u)$ is of the form

$$R(u)=\sum_{n=0}^{[u/a]}(-1)^n\beta^n\,\mathrm{e}^{-na\beta}(u-na)^n/n! \tag{16}$$

The function $R(u)$ can be written more explicitly, step-by-step, for any collision interval:

for $0\leqslant u<a$ $\quad R(u)=1$

for $a\leqslant u<2a$ $\quad R(u)=1-\beta\,\mathrm{e}^{-a\beta}(u-a)$

for $2a\leqslant u<3a$ $\quad R(u)=1-\beta\,\mathrm{e}^{-a\beta}(u-a)+\beta^2\,\mathrm{e}^{-2a\beta}(u-2a)^2/2!$

⋮

for $ka\leqslant u<(k+1)a$ $\quad R(u)=\sum_{n=0}^{n=k}(-1)^n\beta^n\,\mathrm{e}^{-na\beta}(u-na)^n/n! \qquad (17)$

Since $[u/a]$ denotes the greatest integer less than or equal to u/a, and consequently for $ka\leqslant u<(k+1)a, [u/a]=k$, $k=0,1,2,\ldots$. In this way, we can determine, step-by-step, the exact function $f_0(u)$, for any collision interval.

Examining this scheme, in the limiting case of $u\to ka$, it can be seen that

$$\lim_{u\to ka}\sum_{n=0}^{[u/a]}\partial/\partial u(\mathscr{B}^n(u-na)^n/n!)=\lim_{u\to ka}\partial/\partial u\left(\sum_{n=0}^{[u/a]}\mathscr{B}^n(u-na)^n/n!\right) \tag{18}$$

where

$$\mathscr{B}=-\beta\,\mathrm{e}^{-a\beta}. \tag{19}$$

Since, for $u\to ka$, we have

$$\sum_{n=0}^{[u/a]}\partial/\partial u(B^n(u-na)^n/n!)=\partial/\partial u\left(\sum_{n=0}^{[k]}B^n(u-na)^n/n!\right)$$

because the upper limit of the sum ($[u/a]$) is not a variable used in the derivative, since $[u/a]=k$ for any collision interval $ka\leqslant u<(k+1)a$. Obviously, when $u\to ka$, then $[u/a]=k$, $k=0,1,2,\ldots$.

In this way we find

$$R'(u)=\sum_{n=1}^{[u/a]}(-1)^n\beta^n\,\mathrm{e}^{-na\beta}(u-na)^{n-1}/(n-1)! \tag{20}$$

The function $f_{0\delta}(u)$ is a dissonant (changes its form over successive collision intervals) and discontinuous function (in the $u=a$, $f_0(a^-)-f_0(a^+)=\beta$) which after several initial collision intervals becomes constant; namely, for lethargies $u\geqslant 6a$

$$f_0(u)=1/(1-a\beta)=1/\xi. \tag{21}$$

The differential equation, obtainable by differentiation of the initial model (11) becomes

$$f_0'(u)=\beta f_0(u)-\beta f_0(u-a) \tag{22}$$

where $f_0(u)$ is defined in (15) as

$$f_0(u)=(1+\beta)\,\mathrm{e}^{\beta u}R(u)+\mathrm{e}^{\beta u}R'(u). \tag{23}$$

Thus, from equations (22) and (23) (and from (16)) we find

$$R'(u)=-\beta\,\mathrm{e}^{-a\beta}R(u-a)$$

and

$$R(u)+\beta\,e^{-a\beta}\int_0^{u-a} R(u')\,du'=1. \tag{24}$$

Integral equations (11) and (24) are particular cases of the Fredholm integral equation. Integral equation (11) has a unique, positive, and continuous solution in the form (23), for $u>a$. The uniqueness property can be proved by applying the theorem of Banach (model of the fixed point in matrix space). In a similar way, equation (24) has a unique, positive and continuous solution of the form (16), for $u>0$.

On the other hand, integral equation (11) has, also, a particular solution of the form

$$f_{0p}(u)=1/\xi \tag{25}$$

and the particular solution of the integral equation (24) is

$$R_p(u)=(1/\xi)\,e^{-\beta u}-(a\beta/\xi)\,e^{-(1+\beta)u}. \tag{26}$$

On the basis of the theory of Fredholm integral equations (Jörgens 1982, Fenuö 1984, Presdorf 1988) (i.e. the existence of uniqueness properties for the solution of the integral equation) it may be seen that equation (11) has a unique solution under the condition $a\beta<1$. Consequently, the function $f_0(u)$ is unique, and there exists a value of lethargy u_0 when, for $u>u_0$, $f_0(u)=f_{0p}(u)$ and $R(u)=R_p(u)$. Numerical research suggests that u_0 is $6a$ for different moderators. The condition $a\beta<1$ (or $e^a>a+1$) is always satisfied for all mass numbers.

Finally, for lethargies $u\geqslant 6a$ the function $f_0(u)$ becomes a constant function of the form

$$f_0(u)=1/\xi=1/(1-a\beta)$$

and $R(u)$ becomes an exponential function of the form (26).

4. The multiple iteration method

With successive iteration of equation (7) a solution can be found in the following form

$$F(u)=f_{0\delta}(u)-\int_0^u f_0(u-u')\Psi_a(u')f_{0\delta}(u')\,du'+\int_0^u f_0(u-u')\Psi_a(u')\,du' \times\int_0^{u'} f_0(u'-u'')\Psi_a(u'')F(u'')\,du''. \tag{26'}$$

Continuing the iteration in equation (26′), we find

$$F(u)=f_{0\delta}(u)+\sum_{n=1}^{\infty}(-1)^n I_n(u) \tag{27}$$

where

$$I_1(u)=\int_0^u f_0(u-u')\Psi_a(u')f_{0\delta}(u')\,du'$$

and

$$I_n(u)=\int_0^u f_0(u-u')\Psi_a(u')I_{n-1}(u')\,du' \qquad \text{for } n\geqslant 2. \tag{28}$$

Substituting the expression (3) for $f_{0\delta}(u)$ into equation (27) and integrating, we obtain

$$F(u)=(1-\Psi_a(0))\sum_{n=0}^{\infty}(-1)^n K_n(u)+\delta(u) \tag{29}$$

where

$$\begin{aligned} K_0(u)&=f_0(u)\\ K_1(u)&=\int_0^u f_0(u-u')\Psi_a(u')f_0(u')\,\mathrm{d}u'\\ K_n(u)&=\int_0^u f_0(u-u')\Psi_a(u')K_{n-1}(u')\,\mathrm{d}u' \qquad \text{for } n\geqslant 2. \end{aligned} \tag{30}$$

By using the elementary modification of equation (29), $F(u)$ can be expressed as

$$\begin{aligned} F(u)=\delta(u)+(1-\Psi_a(0))f_0(u)\Big[1&-\int_0^u J(u,u')\,\mathrm{d}u'+\int_0^u J(u,u')\,\mathrm{d}u'\int_0^{u'}J(u',u'')\,\mathrm{d}u''\\ &-\int_0^u J(u,u')\,\mathrm{d}u'\int_0^{u'}J(u',u'')\,\mathrm{d}u''\int_0^{u''}J(u'',u''')\,\mathrm{d}u'''+\ldots\Big] \end{aligned} \tag{31}$$

where

$$J(u,u')=\frac{f_0(u-u')\Psi_a(u')f_0(u')}{f_0(u)}. \tag{32}$$

This is the most general complete expression of the solution derived by the multiple iteration method in the isomorphic model of the process. The well-known function $f_0(u)$ is the result due to Teichmann (1960) by applying the method of the Laplace transform.

5. Determination of the iteration coefficients

To simplify the above equation we introduce from (22)

$$f_0(u)=f_0(u+a)-(1/\beta)f_0'(u+a) \tag{33}$$

into the equation for $K_1(u)$ and integrating by parts we have

$$\begin{aligned} \int_0^u f_0(u-u')\Psi_{af}(u')\,\mathrm{d}u' &=(1/\beta)f_0(a)\Psi_{af}(u)-(1/\beta)f_0(u+a)\Psi_{af}(0)\\ &\quad-(1/\beta)\int_0^u f_0(u+a-u')\Psi_{af1}(u')\,\mathrm{d}u' \end{aligned} \tag{34}$$

where

$$\Psi_{af}(u)=\Psi_a(u)f_0(u) \tag{35}$$

$$\Psi_{af1}(u)=\Psi_{af}'(u)-\beta\Psi_{af}(u). \tag{36}$$

Repeating this substitution and integrating by parts, finally a functional $K_1(u)$ can be obtained in the form

$$\int_0^u f_0(u-u')\Psi_{af}(u')\,du'$$

$$=\sum_{n=0}^{n=5}(-1)^{n+1}[D_n\Psi_{af}^{(n)}(u)-D_{nu}(u)\Psi_{af}^{(n)}(0)]+(1/\xi)\int_0^u \Psi_{af}(u')\,du' \qquad (37)$$

where

$$D_{0u}=\left[(6/\xi)-\sum_{k=1}^{k=6} f_0(u+ka)\right]\Big/\beta$$

$$D_{1u}=\left[(15/\xi)-\sum_{k=2}^{k=6}(k-1)f_0(u+ka)\right]\Big/\beta^2$$

$$D_{2u}=[(20/\xi)-f_0(u+3a)-3f_0(u+4a)-6f_0(u+5a)-10f_0(u+6a)]/\beta^3$$

$$D_{3u}=[(15/\xi)-f_0(u+4a)-4f_0(u+5a)-10f_0(u+6a)]/\beta^4 \qquad (38)$$

$$D_{4u}=[(6/\xi)-f_0(u+5a)-5f_0(u+6a)]/\beta^5$$

$$D_{5u}=[(1/\xi)-f_0(u+6a)]/\beta^6$$

$$D_n=D_{nu} \qquad \text{for } u=0.$$

The coefficient D_n and D_{nu} vanishes when $n>6$ since $f_0(u)=1/\xi$ for $u\geqslant 6a$. Finally equation (30) for functional $K_1(u)$ can be reduced to a simpler form:

$$K_1(u)=\Psi_D(u)+(1/\xi)\int_0^u \Psi_{af}(u')\,du' \qquad (39)$$

where

$$\Psi_D(u)=\sum_{n=1}^{n=5}(-1)^{n+1}[D_n\Psi_{af}^{(n)}(u)-D_{nu}(u)\Psi_{af}^{(n)}(0)]. \qquad (40)$$

This, together with the series determined in (29), constitutes the exact solution of the problem. But the slowing down equation (1) or (7) with energy-dependent cross section has not yet been solved for all magnitudes of absorption, since this exact solution in series form is mathematically quite difficult, and practically, is impossible. The solution required for the estimation of terms of series (29) becomes hopelessly involved.

6. General analytical transcendental scheme

In this note we develop a general transcendental method that is applicable for large absorption and for arbitrary energy-dependent cross sections.

The suggested transcendental method is based on the basic premises: exponential form of the collision density and analytic solvability of the corresponding transcendental equation of order one.

The solution in series form in equation (31) can be formally rewritten as

$$F(u)=(1-\Psi_a(0))f_0(u)\left[1-\int_0^u J_0(u')\,du'+\int_0^u J_0(u')\,du'\int_0^{u'} J_0(u'')\,du''\right.$$
$$\left.-\int_0^u J_0(u')\,du'\int_0^{u'} J_0(u'')\,du''\int_0^{u''} J_0(u''')\,du'''+\ldots\right] \tag{41}$$

where

$$J_0(u')=\frac{f_0(u-u')\Psi_a(u')f_0(u')\,du'}{f_0(u)}.$$

But this is correct only for the first collision interval, since for $0<u<a$

$$f_0(u)=(1+\beta)\,e^{\beta u} \qquad \text{and} \qquad J_0(u')=(1+\beta)\Psi_a(u'). \tag{42}$$

For the first collision interval initial model (1) becomes

$$F(u)=(1+\beta)\int_0^u F(u')\,e^{u'-u}\Psi_s(u')\,du'+\delta(u). \tag{43}$$

Equation (43) is an ordinary differential equation, and has analytical solution of the form

$$F_1(u)=(1-\Psi_a(0))f_0(u)\exp\left(-\int_0^u (1+\beta)\Psi_a(u')\,du'\right) \qquad \text{for } 0<u<a. \tag{44}$$

Comparing equations (41) and (44) we find the relationship

$$\exp\left(-\int_0^u J_0(u')\,du'\right)$$
$$=\left[1-\int_0^u J_0(u')\,du'+\int_0^u J_0(u')\,du'\int_0^{u'} J_0(u'')\,du''\right.$$
$$\left.-\int_0^u J_0(u')\,du'\int_0^{u'} J_0(u'')\,du''\int_0^{u''} J_0(u''')\,du'''+\ldots\right]. \tag{45}$$

Consequently, when $J(u,u')=J_0(u')$, solution (41) becomes

$$F(u)=(1-\Psi_a(0))f_0(u)\exp(-S_0(u))$$

where

$$S_0(u)=\int_0^u J_0(u')\,du'. \tag{46}$$

Again, this is correct only for the first collision interval when $J(u,u')=J_0(u')=(1+\beta)\Psi_a(u')$, and, consequently

$$(J(u,u'))'_u=\partial/\partial u(J(u,u'))=\partial/\partial u(J_0(u'))=0.$$

However, in the general case

$$(J(u,u'))'_u\neq 0.$$

But, of course, we can also assume a general solution in the form,

$$F_\delta(u)=(1-\Psi_a(0))f_0(u)\exp(-\varepsilon_u S(u))+\delta(u) \tag{47}$$

where

$$S(u)=\int_0^u \frac{f_0(u-u')\Psi_a(u')f_0(u')\,du'}{f_0(u)} \tag{48}$$

and ε_u is the dissonant, real constant. Namely, ε_u is the constant in the area Δu around the lethargy u. Thus,

$$\varepsilon_{u+\Delta u}-\varepsilon_{u-\Delta u}=\Delta\varepsilon \tag{49}$$

and the right-hand side of the above equality tends to zero as Δu tends to γ, where γ is an arbitrary small positive, real number, and

$$\gamma \ll a. \tag{50}$$

For the first collision interval

$$\varepsilon_u=1 \qquad \text{for } 0<u<a. \tag{51}$$

The hypothesis (47) is crucial for the transcendental method in the theory of slowing down. But, generally, in equation (47) the collision density is represented in a particular way. That, of course, may or may not be true. It is important that this proposal is permissible.

The suggested solution (47) satisfies the differential equation obtainable by differentiation of the initial model (1):

$$F'_\delta(u)+F_\delta(u)=(1+\beta)\Psi_s(u)F(u)-\beta\Psi_s(u-a)F(u-a)+\delta'(u)+\delta(u) \tag{52}$$

where

$$F_\delta(u)=F(u)+\delta(u) \tag{53}$$

or for $u>0$ (52) becomes

$$F'(u)+F(u)=(1+\beta)\Psi_s(u)F(u)-\beta\Psi_s(u-a)F(u-a). \tag{54}$$

Substituting the result (47) for $F(u)$ in equation (54), we obtain:

$$\frac{f_0'(u)}{f_0(u)}-\varepsilon_u S'(u)+1=(1+\beta)\Psi_s(u)-\beta\Psi_s(u-a)\frac{f_0(u-a)}{f_0(u)}\exp(Z(u)) \tag{55}$$

where

$$Z(u)=\varepsilon_u S(u)-\varepsilon_{u-a}S(u-a) \tag{56}$$

and from the above equation we have

$$\varepsilon_u=\frac{Z(u)+\varepsilon_{u-a}S(u-a)}{S(u)}. \tag{57}$$

From equation (22)

$$(f_0'(u)/f_0(u))=\beta-\beta f_0(u-a)/f_0(u). \tag{58}$$

Substituting the above result in the left-hand side of equation (55) we obtain

$$\begin{aligned}(1+\beta)\Psi_a(u)-(\beta f_0(u-a)/f_0(u))-S'(u)\varepsilon_{u-a}S(u-a)/S(u)\\ +\beta\Psi_s(u-a)(f_0(u-a)/f_0(u))\exp(Z(u))\\ =Z(u)S'(u)/S(u).\end{aligned} \tag{59}$$

To simplify the above equation, introduce

$$A_0(u)=S'(u)/S(u)$$
$$A_1(u)=(1+\beta)\Psi_a(u)-A_0(u)S(u-a)\varepsilon_{u-a}-\beta f_0(u-a)/f_0(u)$$
$$A_2(u)=\beta\Psi_s(u-a)f_0(u-a)/f_0(u)$$

where

$$S'(u)=(1+\beta)\Psi_a(u)+(\beta f_0(u-a)/f_0(u))[S(u)-\Psi_a(u-a)-S(u-a)]$$

into equation (59) and find

$$A_1(u)+A_2(u)\exp(Z(u))=A_0(u)Z(u). \tag{60}$$

For the first collision interval $(0<u<a)$, $\varepsilon_u=1$.

After elementary modification, the transcendental equation which the required $Z(u)$ should satisfy becomes

$$Z_1(u)=B_1(u)\exp(B_0(u)Z_1(u)) \tag{61}$$

where

$$B_0(u)=1/A_0(u) \qquad B_1(u)=A_2(u)\exp(A_1(u)/A_0(u)) \tag{62}$$

and

$$Z(u)=(Z_1(u)+A_1(u))B_0(u). \tag{63}$$

7. The final form of the solution for collision density

Finally, it is possible to get the solution (47) to the slowing down equation (1) with energy-dependent cross sections written in the form

$$F(u)=\frac{f_0(u)}{f_0(u-a)}F(u-a)\exp(-Z(u)) \qquad \text{for } u>0 \tag{64}$$

or, repeating the substitution method for lethargies $na<u<(n+1)a$, the collision density has the form

$$F(u)=\frac{f_0(u)}{f_0(u-na)}F_1(u-na)\exp\left(-\sum_{n=0}^{[(u-a)/a]}Z(u-na)\right) \tag{65}$$

where $F_1(u-na)$ is the well-known solution for the first collision interval. The method of obtaining $F_1(u)$ is given in section 6 in equation (44), namely,

$$F_1(u-na)=(1+\beta)(1-\Psi_a(0))\exp(\beta(u-na)-(1+\beta)I(u-na)) \tag{66}$$

where

$$I(u-na)=\int_0^{u-na}\Psi_a(u')\,du'$$

and $Z(u-na)$ is the analytical elementary solution of the transcendental equation (61) (see section 8).

Equation (65) is the exact analytical closed-form solution to the slowing down equation with energy-dependent cross section for heavy nuclei and for all magnitudes of absorption. The result (65) is very important because it is not mathematically difficult.

The solution (65), together with the exact solution of the transcendental equation determined in (61), constitutes the exact solution of the problem for arbitrary energy-dependent cross section and for different moderators.

The method for obtaining the analytical solution of the transcendental equation (61) is given in Perovich (1990, 1991) and also in the next section.

8. The analytical solution of the transcendental equation (61)

The transcendental equation (61) can be identified with an integral equation of type

$$\Phi(y) = B_1(u) \int_{y-B_0(u)}^{\infty} \Phi(t)\,\mathrm{d}t \tag{67}$$

where $\Phi(y)$ is an arbitrary real function and $y > B_0(u)$.

This integral equation is analytically solvable using a Laplace transform. The analytical unique solution (Perovich 1991) is the following series:

$$\Phi(y) = \Phi_0 \sum_{n=0}^{[y/B_0(u)]} (-1)^n B_1^n(u)(y - nB_0(u))^n / n! \tag{68}$$

On the other hand, the integral equation (67) has a particular solution in the form

$$\Phi_{\rm p}(y) = \Phi_{\rm p0} \exp(-Z_1(u)y) \tag{69}$$

which satisfies equation (67) if and only if $Z_1(u)$ satisfies the transcendental equation

$$Z_1(u) = B_1(u) \exp(B_0(u)Z_1(u)). \tag{70}$$

Finally, for the value $y > y_0$ (since for $y > y_0$ $\Phi(y) = \Phi_{\rm p}(y)$) it is possible to establish the equality

$$\Phi(y - B_0(u))/\Phi(y) = \exp(B_0(u)Z_1(u)) \tag{71}$$

or

$$Z_1(u) = (1/B_0(u)) \ln[\Phi(y - B_0(u))/\Phi(y)] \tag{72}$$

and from equations (63) and (72) we obtain

$$Z(u) = (A_1(u)/A_0(u)) + \ln[\Phi(y - B_0(u))/\Phi(y)] \tag{73}$$

where the function $\Phi(y)$ given in (68) can be rewritten as

$$\Phi(B_0(u)y) = \Phi_0 \sum_{n=0}^{[y]} (-1)^n (B_1(u)B_0(u))^n (y-n)^n / n! \tag{74}$$

and

$$Z(u) = (A_1(u)/A_0(u)) + \ln[\phi(B_0(u)(y-1))/\Phi(B_0(u)y)]. \tag{75}$$

9. Slowing down with constant cross section

In this section we present a complete, exact transcendental theory of neutron slowing down in a single element with isotropic elastic scattering in the centre-of-system and with constant cross section.

In this case the isomorphic model of the process (7) becomes

$$F(u)=f_{0\delta}(u)-c_0\int_0^u f_0(u-u')F(u')\,du' \tag{76}$$

where $\Psi_a(u)=c_0=$ constant is the absorption rate.

The integral equation (76) is analytically solvable. Namely, applying the method of the Laplace transform on integral equation (76), it is possible to obtain an analytical solution of the form

$$F_c(u)=(bR_0(u)+R_0'(u))\exp((b-1)u) \tag{77}$$

where $b=(1+\beta)(1-c_0)$ and

$$R_0(u)=\sum_{n=0}^{[u/a]}(-1)^n b^n\,e^{-nab}(u-na)^n/n! \tag{78}$$

where $[u/a]$ denotes the greatest integer less than or equal to u/a.

The solution (77) is the Placzek function obtained from the Teichmann series.

But the integral equation (76) can also be analytically solved by application of the transcendental method. In this case, for lethargies $a<u<2a$, the transcendental model (61) can be rewritten as

$$Z_1(u)=B_1(u)\exp(B_0(u)Z_1(u)) \tag{79}$$

where

$$\begin{aligned}
S(u)&=((1+\beta)\,e^{\beta u}c_0/f_0(u))[-\beta\,e^{-a\beta}(u-a)^2-2\alpha\,e^{-a\beta}(u-a)+u]\\
S(u-a)&=(1+\beta)c_0(u-a)\\
B_0(u)&=S(u)/S'(u)
\end{aligned} \tag{80}$$

where

$$S'(u)=(1+\beta)c_0+(\beta f_0(u-a)/f_0(u))[S(u)-c_0-S(u-a)]$$

where the function $f_0(u)$ is defined in equations (4) and (14).

The coefficient $B_1(u)$ is

$$B_1(u)=\beta(1-c_0)(f_0(u-a)/f_0(u))\exp(-S(u-a)+C_\beta S(u)/S'(u)) \tag{81}$$

where $C_\beta=(1+\beta)c_0-\beta f_0(u-a)/f_0(u)$.

For the second collision interval $\varepsilon_{u-a}=1$.

Having obtained the exact value of $Z_1(u)$ from equation (79) we now proceed to develop a general scheme for obtaining an analytical closed-form solution.

Now, the general transcendental scheme for the second collision interval becomes

$$F_2(u)=\frac{f_0(u)}{f_0(u-a)}\,b\,e^{(b-1)(u-a)}\exp(-Z_2(u)). \tag{82}$$

Of course, in the lethargy interval $a<u<2a$, $Z_1(u)$ is the exact solution of the transcendental equation (79) and

$$Z_2(u)=(Z_1(u)+A_1(u))B_0(u) \tag{83}$$

where

$$A_1(u)=(1+\beta)c_0-\beta f_0(u-a)/f_0(u)-S(u-a)S(u)/S'(u)$$

Table 1. Comparisons between Placzek (Teichmann) and present solution.

A	c_0	u/a	$F(u)$ Equation (77) $F_c(u)$	Equation (82) $F_2(u)$	Equation (94) $G_0(u)$
2	0.0001	1.1	1.361029	1.361029	−1.19602E−07
		1.3	1.370725	1.370725	6.47645E−07
		1.5	1.377184	1.377184	7.00167E−07
		1.7	1.380009	1.380010	−3.05868E−06
		1.9	1.378770	1.378777	−2.94455E−06
12	0.0001	1.2	5.986350	5.98630	−3.70529E−06
		1.4	6.254024	6.25400	−6.25670E−07
		1.6	6.418465	6.41850	2.46440E−06
		1.8	6.432793	6.43300	3.68649E−06
		2.0	6.236403	6.23650	1.23932E−05
50	0.0001	1.2	23.59868	23.59870	−5.71954E−07
		1.4	24.92845	24.92852	5.08553E−06
		1.6	25.76308	25.76300	7.48569E−05
		1.8	25.83322	25.83311	3.37631E−07
		2.0	24.77766	25.7801	8.46375E−05
A	c_0	u/a	$F_c(u)$	Equation (84) $F_3(u)$	$G_0(u)$
2	0.0001	2.2	1.377302	1.37730	−5.45507E−06
		2.4	1.377621	1.37762	−5.48874E−06
		2.6	1.377593	1.37761	−5.33532E−06
		2.8	1.377404	1.37740	−5.01503E−06
12	0.0001	2.2	6.313301	6.31330	3.20922E−07
		2.4	6.347757	6.34776	1.77588E−06
		2.6	6.348347	6.34835	2.99369E−06
		2.8	6.331448	6.33145	3.14782E−06
		3.0	6.326562	6.32650	3.11351E−05
50	0.0001	2.2	25.20122	25.2012	5.73112E−06
		2.4	25.39747	25.3975	1.26634E−06
		2.6	25.40037	25.4001	1.52236E−05
		2.8	25.30001	25.300	1.60032E−05
A	c_0	u/a	$F_c(u)$	Equation (86) $F_4(u)$	$G_0(u)$
2	0.0001	3.2	1.377231	1.37710	−5.01809E−07
		3.4	1.377154	1.37716	−1.64460E−06
		3.6	1.377065	1.37707	−3.03511E−07
		3.8	1.376978	1.376989	5.75688E−07
12	0.0001	3.2	6.335561	6.33550	−9.46459E−05
		3.3	6.336478	6.33650	−1.85559E−06
		3.5	6.334671	6.33465	−9.82439E−06
		3.7	6.332228	6.33223	6.14402E−06
		3.9	6.332200	6.33200	5.68803E−06
50	0.0001	3.2	25.32868	25.330	−1.59475E−05
		3.4	25.33166	25.331	−2.91189E−05
		3.6	25.31454	25.315	−1.13752E−06
		3.8	25.30598	25.306	1.52994E−05

Table 1. (continued)

			$F(u)$		
A	c_0	u/a	Equation (77) $F_c(u)$	Equation (85) $F_2(u)$	Equation (94) $G_0(u)$
2	0.0001	4.2	1.376814	1.37678	−5.01809E−08
		4.4	1.376730	1.376770	−1.64466E−06
		4.7	1.376605	1.37663	1.73344E−07
		4.9	1.376522	1.37650	3.81969E−07
12	0.0001	4.0	6.332825	6.33278	2.68072E−06
		4.3	6.332643	6.33260	−1.85559E−06
		4.5	6.332012	6.3320	−9.82444E−06
		4.7	6.331713	6.33167	6.14402E−06
		4.9	6.331621	6.33157	5.68803E−05
50	0.0001	4.2	25.31528	25.315	−1.59475E−05
		4.4	25.31143	25.311	−2.91189E−05
		4.6	25.30880	25.309	−1.13752E−06
		4.8	25.30812	25.308	1.52994E−05

and from (72)

$$Z_1(u) = (1/B_0(u))\ln(\Phi(y-B_0(u))/\phi(y))$$

where

$$\Phi(y) = \Phi_0 \sum_{n=0}^{[y/B_0(u)]} (-1)^n B_1^n(u)(u-nB_0(u))^n/n!$$

and $b\,\mathrm{e}^{(b-1)(u-a)}$ is the analytical solution for the collision density on the first collision interval. Applying this general transcendental scheme for the lethargies in the interval $2a<u<3a$, a solution $F_3(u)$ can be obtained in the form

$$F_3(u) = \frac{f_0(u)}{f_0(u-a)} F_2(u-a)\exp(-Z_3(u)) \tag{84}$$

where $Z_3(u)$ is the analytical elementary solution of the transcendental equation (79) with new values for $B_0(u)$ and $B_1(u)$, and for

$$^2\varepsilon_{u-a} = \frac{Z_2(u)+S(u-2a)}{S(u-a)}. \tag{85}$$

Further, continuing the similar reasoning for the next collision interval $3a<u<4a$ we obtain a collision density in the form

$$F_4(u) = \frac{f_0(u)}{f_0(u-a)} F_3(u-a)\exp(-Z_4(u)) \tag{86}$$

where

$$^3\varepsilon_{u-a} = \frac{Z_3(u-a)+S(u-2a)^2\varepsilon_{u-a}}{S(u-a)}$$

etc.

Table 2.

A	c_0	u/a	$F_c(u)$	Equation (82) $F_2(u)$	$G_0(u)$
2	0.1	1.2	0.882975	0.8830	−2.68218E−07
		1.4	0.837231	0.8371	−3.68800E−06
		1.6	0.790955	0.7908	−7.66670E−06
		1.8	0.744144	0.7442	−1.25711E−05
12	0.1	1.2	4.382013	4.3821	7.33089E−06
		1.4	4.423473	4.4235	3.97056E−06
		1.6	4.371296	4.3710	−1.23758E−05
		1.8	4.195269	4.1953	1.10312E−05
		2.0	3.858035	3.85801	−3.29814E−05
50	0.1	1.2	17.487062	17.4871	3.27544E−05
		1.4	17.903863	17.9041	1.98063E−06
		1.6	17.865946	17.8658	−1.54311E−06
		1.8	17.188849	17.1887	−4.55993E−06
		2.0	15.635042	15.6348	−1.58223E−05

A	c_0	u/a	$F_c(u)$	Equation (84) $F_3(u)$	$G_0(u)$
2	0.1	2.2	0.655735	0.6563	−5.21659E−05
		2.4	0.616706	0.6188	−5.15841E−05
		2.6	0.579742	0.5810	−4.90777E−05
		2.8	0.544882	0.5458	−4.47676E−05
12	0.1	2.2	3.753327	3.7532	4.09261E−06
		2.4	3.619316	3.6191	1.11307E−05
		2.6	3.464796	3.4650	1.08875E−05
		2.8	3.304094	3.3041	−1.45535E−05
		3.0	3.159317	3.1593	−1.29318E−06
50	0.1	2.2	15.325251	15.3253	4.19387E−06
		2.4	14.846722	14.8468	7.69482E−06
		2.6	14.239304	14.2389	6.02365E−06
		2.8	13.580812	13.5810	1.81955E−05
		3.0	13.003082	13.0031	−0.47851E−05

A	c_0	u/a	$F_c(u)$	Equation (86) $F_4(u)$	$G_0(u)$
2	0.1	3.2	0.481467	0.4810	−6.34351E−07
		3.4	0.452587	0.4526	−3.31954E−06
		3.6	0.425432	0.4255	−6.01569E−07
		3.8	0.399907	0.3999	1.10312E−06
12	0.1	3.2	3.030517	3.0304	−5.14896E−06
		3.4	2.901131	2.9021	−9.37119E−05
		3.6	2.775209	2.7750	−3.90287E−06
		3.8	2.655589	2.6553	4.90520E−06
50	0.1	3.2	12.509593	12.5120	3.92745E−06
		3.4	11.998172	11.9982	7.36484E−06
		3.6	11.494123	11.4938	3.13228E−07
		3.8	11.016724	11.0159	−3.93677E−06
		3.8	11.016724	11.0159	−3.93677E−06
		4.0	10.569499	10.5695	−1.03449E−07

Table 2. (continued)

A	c_0	u/a	$F_c(u)$	Equation (88) $F_5(u)$	$G_0(u)$
2	0.1	4.2	0.353366	0.3532	−6.34351E−07
		4.4	0.332167	0.33218	−3.31954E−06
		4.6	0.312239	0.31228	−6.01569E−07
		4.8	0.293508	0.29351	1.10312E−07
12	0.1	4.0	2.542631	2.5400	1.18661E−06
		4.3	2.381144	2.3801	−1.00614E−06
		4.5	2.278919	2.2800	−5.39647E−06
		4.7	2.181279	2.1813	3.52100E−06
		4.9	2.087944	2.0878	3.28426E−06
50	0.1	4.2	10.136834	10.13679	3.92745E−06
		4.4	9.718432	9.7291	7.36484E−06
		4.6	9.318020	9.3179	3.13228E−06
		4.8	8.935602	8.9346	−3.93676E−06
		5.0	8.568878	8.5689	−1.03449E−07

For the lethargies $u > 4a$ the equation (55) becomes

$$Z_0 = a\beta(1-c_0)(e^{Z_0}-1) + ac_0 \tag{87}$$

since, for $u > 4af_0(u) \approx 1/\xi$, and $Z(u) = \text{constant} = Z_0$.

Thus, the analytical solution for the collision density on the next collision intervals has the following forms:

$$F_5(u) = F_4(u)\exp(-Z_0) \tag{88}$$

and, also

$$F_n(u) = F_4(u-(n-4)a)\exp(-(n-4)Z_0) \qquad \text{for } n > 6 \tag{89}$$

where, of course, Z_0 is the exact solution of the transcendental equation (87).

The transcendental equation (87) can be rewritten as

$$Z_{01} = B_c \exp(Z_{01}) \tag{90}$$

where $B_c = a\beta(1-c_0)\exp(ac_0 - a\beta(1-c_0))$. From section 8:

$$Z_{01} = \ln(\Phi_0(y-1)/\Phi_0(y)) \tag{91}$$

where

$$\Phi_0(y) = \Phi_{00} \sum_{n=0}^{[y]} (-1)^n B_c^n (u-n)^n / n! \tag{92}$$

and

$$Z_0 = Z_{01} + ac_0 - a\beta(1-c_0). \tag{93}$$

The results in this case of simple energy-dependent cross section (ideally constant) are identical with the results obtained with another method (Teichmann 1961, Sengupta *et al* 1974). It suggests that the proposed transcendental method has validity.

The error function is defined from equation (61) as

$$G_0(u) = Z_1(u) - B_1(u)\exp(B_0(u)Z_1(u)). \tag{94}$$

Table 3.

A	c_0	u/a	$F_c(u)$	Equation (82) $F_2(u)$	$G_0(u)$
2	0.25	1.2	0.438864	0.4388	6.99875E−06
		1.4	0.379441	0.3794	−2.03069E−06
		1.6	0.325972	0.3260	−5.26443E−06
		1.8	0.277929	0.2781	−9.37163E−06
12	0.25	1.2	2.615773	2.6157	5.49552E−05
		1.4	2.503461	2.5031	3.14964E−05
		1.6	2.330008	2.3300	−5.77615E−05
		1.8	2.082753	2.0830	−7.93326E−05
		2.0	1.746924	1.7473	−2.57610E−05
50	0.25	1.2	10.629728	10.6301	0.24245E−05
		1.4	10.368355	10.3683	0.15313E−05
		1.6	9.788049	9.7881	2.28463E−06
		1.8	8.798805	8.7988	−0.32486E−05
		2.0	7.290762	7.2908	−0.12348E−05

A	c_0	u/a	$F_c(u)$	Equation (84) $F_3(u)$	$G_0(u)$
2	0.25	2.2	0.200494	0.2005	−5.99220E−06
		2.4	0.170933	0.1712	−2.49027E−06
		2.6	0.145578	0.1455	−2.58365E−06
		2.8	0.123923	0.1250	1.34390E−05
12	0.25	2.2	1.595353	1.5952	−6.20650E−05
		2.4	1.437684	1.4377	−7.49828E−06
		2.6	1.280621	1.2805	−2.44098E−05
		2.8	1.133204	1.1330	−7.42155E−05
		3.0	1.007348	1.0072	−1.50848E−05
50	0.25	2.2	6.739050	6.7391	0.36873E−05
		2.4	6.124525	6.1243	0.24768E−05
		2.6	5.481063	5.4810	−0.10650E−05
		2.8	4.860204	4.8600	−0.35899E−05
		3.0	4.336608	4.3365	−0.73722E−05

A	c_0	u/a	$F_c(u)$	Equation (86) $F_4(u)$	$G_0(u)$
2	0.25	3.2	0.089874	0.0898	−3.46973E−07
		3.4	0.076537	0.07655	−9.04277E−06
		3.6	0.065176	0.06520	−1.16750E−06
		3.8	0.055026	0.05503	3.50401E−06
12	0.25	3.2	0.900651	0.8998	2.18448E−06
		3.4	0.802291	0.8019	4.16995E−06
		3.6	0.713635	0.7128	1.93367E−06
		3.8	0.635159	0.6351	−2.29489E−06
		4.0	0.566013	0.5657	−6.71612E−06
50	0.25	3.2	3.901799	3.9018	2.59127E−06
		3.4	3.492021	3.4920	5.24942E−05
		3.6	3.118392	3.1184	2.93498E−06
		3.8	2.787192	2.7872	−3.01056E−06
		4.0	2.496003	2.4960	−1.21479E−07

Table 3. (continued)

A	c_0	u/a	$F_c(u)$	Equation (88) $F_2(u)$	$G_0(u)$
2	0.25	4.2	0.040252	0.040216	−3.46973E−06
		4.4	0.034278	0.034285	−9.04277E−06
		4.6	0.029191	0.029120	−1.16750E−06
		4.8	0.024858	0.024858	3.50400E−06
		5.0	0.021169	0.021166	3.74765E−07
12	0.25	4.3	0.475726	0.4747	4.38747E−06
		4.4	0.448877	0.4491	4.16995E−06
		4.6	0.399688	0.4001	1.93367E−07
		4.8	0.355968	0.3559	−2.29489E−06
		5.0	0.317030	0.3168	−6.71612E−07
50	0.25	4.2	2.233701	2.2340	2.59127E−06
		4.4	1.997468	1.9975	5.24942E−06
		4.6	1.786488	1.7865	2.93498E−06
		4.8	1.598392	1.5984	−3.01056E−06
		5.0	1.430114	1.4300	−1.21479E−07

The calculation has been done on an ordinary PC.

Some calculations based on equations (82), (84), (86), (88) and (94) are shown in tables 1, 2 and 3 for various values of c_0 and for mass number $A=2$, $A=12$ and $A=50$ in the region $a<u<5a$ for comparison with the Teichmann series (77).

In the numerical calculations the sum $\Phi(B_0(u)y)$ in equation (74) was terminated after $N=10$ (or $N=15$) or when $G_0<10^{-4}$. But the rather high values of N for large u/a (relative lethargy) depend on the limit equation (94).

For practical calculations, of course, the number of terms in the series $\Phi(y)$ can be reduced considerably.

Obviously, the transcendental scheme gives an explicit expression for the nth collision density (equation (65)).

But the usefulness of this transcendental approach depends, of course, on the case with which reasonably accurate solutions of the transcendental equation (61) may be obtained, especially for large absorption rates and for large mass numbers.

The solution (65) has been calculated explicitly and exists as a satisfactory solution in the numerical sense, if and only if $G_0(u)$ (in (94)) satisfies the inequality

$$G_0(u)<10^{-4}.$$

With high-speed computers, this method does not appear to be restrictive.

10. Conclusions

Difficulties in obtaining analytical solutions to the neutron slowing down equation arise when the cross sections are energy-dependent.

The present paper removes this difficulty and the slowing down equation with energy-dependent cross section has been analytically solved.

Thus, a general transcendental scheme for obtaining an analytic closed-form solution is derived. According to the author's knowledge, this is the first application of a direct transcendental method to the slowing down problem with energy-dependent cross section.

The method will provide accurate solutions and is generalizable to more comprehensive problem such as mixtures. A detailed study of the transcendental method for mixtures will be the subject of another publication.

The suggested transcendental scheme can be routinely applied to energy-dependent cross section as well as to constant cross section problems.

The given numerical transcendental method package for the evaluation of the collision density is essentially reduced to a transcendental equation calculation.

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